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Three-step iterative algorithms for solving the system of generalized mixed quasi-variational-like inclusions

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Abstract

In this paper, we consider the system of generalized mixed quasi-variational-like inclusions in Hilbert spaces. We extend the auxiliary principle technique to develop a three-step iterative algorithm for solving the system of generalized mixed quasi-variational-like inclusions. Under the assumptions of the continuity and partially relaxed η-strong monotonicity of set-valued mappings, we establish the convergence for our algorithm. Our algorithm and its convergence results are new, and generalize Ding's predictor–corrector iterative algorithms. Moreover, our results unify some known results in the literature as well.

Keywords: Generalized mixed quasi-variational-like inclusion; Auxiliary principle technique; Partially relaxed η-strongly monotone mappings; Three-step iterative algorithm; Set-valued mapping

1. Introduction

In recent years, one of the most significant and important problems in the variational inequality theory is the development of efficient iterative algorithms to compute approximate solutions. Although one of the most effective numerical techniques for variational inequalities is the projection method and its variant forms, the standard projection technique cannot be applied for general mixed variational inequalities directly. To improve this situation, several authors have developed the auxiliary principle technique to study the existence and iterative algorithm of solutions for various nonlinear mixed variational (variational-like) inequalities (e.g. see, [1–9]).

Recently, Noor [10–12] has introduced a new class of predictor–corrector iterative algorithms for solving general mixed variational inequalities. By applying the auxiliary principle technique, he tried to prove the convergence of the iterative sequences generated by his predictor–corrector algorithms. As pointed out in Ding [18,19], it appears that there are some shortcomings in the proof of Lemma 3.1 and Theorem 3.1 in [10–12]. Hence, it became an open question as to how to use the predictor–corrector-type iterative algorithms for solving generalized mixed variational-like inequality problems. Ding [13] proved the convergence of the predictor–corrector iterative algorithm for solving

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nonlinear mixed variational-like inequalities. Moreover, Ding [18,19] gave a complete answer to this difficult problem. In [18], he introduced a concept of partially relaxed η-strong monotonicity for a set-valued mapping. In [19], by applying the concept and auxiliary principle technique, he suggested some predictor–corrector iterative schemes for solving generalized mixed quasi-variational-like inclusions. His convergence analysis requires only the continuity and partially relaxed η-strong monotonicity for the underlying mappings.

Motivated and inspired by Ding [19], we introduce and consider the system of generalized mixed quasi-variational-like inclusions in Hilbert spaces. The auxiliary principle technique is extended to develop a three-step iterative algorithm for solving the system of generalized mixed quasi-variational-like inclusions. Under similar assumptions of the continuity and partially relaxed η-strong monotonicity of set-valued mappings, we prove the convergence using our algorithm.

The paper is organized as follows. Section 2 contains some preliminary settings. In Section 3, we propose the three-step iterative algorithm and establish its convergence. We then give a detailed illustration on how our results extend and unify some known results [10–13,18,19] in Section 4.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. Let $CB(H)$ be the family of all nonempty bounded closed subsets of H. Let $\tilde{H}(\cdot,\cdot)$ be the Hausdorff metric on $CB(H)$ defined by

$$\tilde{H}(A,B)=\max\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(A,y)\right\}, \quad \forall A,B\in CB(H).$$

Let $T,\hat{T},\bar{T},A,\hat{A},\bar{A}:H\to CB(H)$ be set-valued mappings. Let $N,\hat{N},\bar{N},\eta,\hat{\eta},\bar{\eta}:H\times H\to H$ be single-valued mappings, and $\varphi,\hat{\varphi},\bar{\varphi}:H\times H\to(-\infty,+\infty]$ be real bifunctions. We consider the system of generalized mixed quasi-variational-like inclusions (in short, SGMQVLI): find $x\in H, u\in T(x), \hat{u}\in\hat{T}(x), \bar{u}\in\bar{T}(x), v\in A(x), \hat{v}\in\hat{A}(x)$ and $\bar{v}\in\bar{A}(x)$ such that

$$\begin{cases}\langle N(u,v),\eta(y,x)\rangle+\varphi(y,x)-\varphi(x,x)\ge 0, & \forall y\in H,\\ \langle \hat{N}(\hat{u},\hat{v}),\hat{\eta}(y,x)\rangle+\hat{\varphi}(y,x)-\hat{\varphi}(x,x)\ge 0, & \forall y\in H,\\ \langle \bar{N}(\bar{u},\bar{v}),\bar{\eta}(y,x)\rangle+\bar{\varphi}(y,x)-\bar{\varphi}(x,x)\ge 0, & \forall y\in H.\end{cases} \tag{2.1}$$

2.1. Special cases

(I) If the bifunctions $\varphi(\cdot,\cdot)$, $\hat{\varphi}(\cdot,\cdot)$ and $\bar{\varphi}(\cdot,\cdot)$ are η-subdifferentiable, $\hat{\eta}$-subdifferentiable and $\bar{\eta}$-subdifferentiable in the first argument, respectively, and are all lower semicontinuous in the first argument, then the SGMQVLI (2.1) reduces to the following system of variational inclusions: find $x\in H, u\in T(x), \hat{u}\in\hat{T}(x), \bar{u}\in\bar{T}(x), v\in A(x), \hat{v}\in\hat{A}(x)$ and $\bar{v}\in\bar{A}(x)$ such that

$$\begin{cases}0\in N(u,v)+\Delta\varphi(x,x),\\ 0\in \hat{N}(\hat{u},\hat{v})+\Delta\hat{\varphi}(x,x),\\ 0\in \bar{N}(\bar{u},\bar{v})+\Delta\bar{\varphi}(x,x),\end{cases} \tag{2.1$'$}$$

where for each $y\in H$, $\Delta\varphi(x,y)$, $\Delta\hat{\varphi}(x,y)$ and $\Delta\bar{\varphi}(x,y)$ denote the η-subdifferential of $\varphi(\cdot,y)$ at x, the $\hat{\eta}$-subdifferential of $\hat{\varphi}(\cdot,y)$ at x and the $\bar{\eta}$-subdifferential of $\bar{\varphi}(\cdot,y)$ at x, respectively; see [21].

(II) If $\eta(x,y)=g(x)-g(y)$, $\hat{\eta}(x,y)=\hat{g}(x)-\hat{g}(y)$, and $\bar{\eta}(x,y)=\bar{g}(x)-\bar{g}(y)$, $\forall x,y\in H$ where $g,\hat{g},\bar{g}:H\to H$ are three given single-valued mappings, then the SGMQVLI (2.1) reduces to the following system of generalized mixed quasi-variational inclusions (in short, $SGMQVI$): find $x\in H, u\in T(x), \hat{u}\in\hat{T}(x), \bar{u}\in\bar{T}(x), v\in A(x), \hat{v}\in\hat{A}(x)$ and $\bar{v}\in\bar{A}(x)$ such that

$$\begin{cases}\langle N(u,v),g(y)-g(x)\rangle+\varphi(y,x)-\varphi(x,x)\ge 0, & \forall y\in H,\\ \langle \hat{N}(\hat{u},\hat{v}),\hat{g}(y)-\hat{g}(x)\rangle+\hat{\varphi}(y,x)-\hat{\varphi}(x,x)\ge 0, & \forall y\in H,\\ \langle \bar{N}(\bar{u},\bar{v}),\bar{g}(y)-\bar{g}(x)\rangle+\bar{\varphi}(y,x)-\bar{\varphi}(x,x)\ge 0, & \forall y\in H.\end{cases} \tag{2.2}$$

(III) If $\eta(x,y)=\hat{\eta}(x,y)=\bar{\eta}(x,y)=x-y, \forall x,y\in H$, then the SGMQVLI (2.1) reduces to the following system of generalized mixed quasi-variational inclusions: find $x\in H, u\in T(x), \hat{u}\in\hat{T}(x), \bar{u}\in\bar{T}(x), v\in A(x), \hat{v}\in\hat{A}(x)$ and $\bar{v}\in\bar{A}(x)$ such that

$$\begin{cases}\langle N(u,v), y-x\rangle+\varphi(y,x)-\varphi(x,x)\geq 0, & \forall y\in H,\\ \langle\hat{N}(\hat{u},\hat{v}), y-x\rangle+\hat{\varphi}(y,x)-\hat{\varphi}(x,x)\geq 0, & \forall y\in H,\\ \langle\bar{N}(\bar{u},\bar{v}), y-x\rangle+\bar{\varphi}(y,x)-\bar{\varphi}(x,x)\geq 0, & \forall y\in H.\end{cases} \tag{2.3}$$

(IV) If $N(u,v)=\hat{N}(u,v)=\bar{N}(u,v)=u-v, \forall u,v\in H$, then the SGMQVLI (2.1) reduces to the following system of generalized mixed quasi-variational-like inclusions: find $x\in H, u\in T(x), \hat{u}\in\hat{T}(x), \bar{u}\in\bar{T}(x), v\in A(x), \hat{v}\in\hat{A}(x)$ and $\bar{v}\in\bar{A}(x)$ such that

$$\begin{cases}\langle u-v, \eta(y,x)\rangle+\varphi(y,x)-\varphi(x,x)\geq 0, & \forall y\in H,\\ \langle\hat{u}-\hat{v}, \hat{\eta}(y,x)\rangle+\hat{\varphi}(y,x)-\hat{\varphi}(x,x)\geq 0, & \forall y\in H,\\ \langle\bar{u}-\bar{v}, \bar{\eta}(y,x)\rangle+\bar{\varphi}(y,x)-\bar{\varphi}(x,x)\geq 0, & \forall y\in H.\end{cases} \tag{2.4}$$

(V) If $T=\hat{T}=\bar{T},\ A=\hat{A}=\bar{A},\ N=\hat{N}=\bar{N}, \eta=\hat{\eta}=\bar{\eta}$, and $\varphi=\hat{\varphi}=\bar{\varphi}$, then the SGMQVLI (2.1) reduces to the following generalized mixed quasi-variational-like inclusion problem (in short, GMQVLIP): find $x\in H, u\in T(x)$ and $v\in A(x)$ such that

$$\langle N(u,v), \eta(y,x)\rangle+\varphi(y,x)-\varphi(x,x)\geq 0, \quad \forall y\in H. \tag{2.5}$$

The problem (2.5) was introduced and considered by Ding [19] which includes a number of extensions and generalizations of generalized variational and variational-like inequalities in the literature as special cases, see [1–14,18] and the references therein.

Definition 2.1. The bifunction $\varphi(\cdot,\cdot)$ is said to be skew-symmetric if

$$\varphi(x,x)-\varphi(x,y)-\varphi(y,x)+\varphi(y,y)\geq 0, \quad \forall x,y\in H.$$

The skew-symmetric bifunctions have the properties analogous to the monotonicity of the gradient and the nonnegativity of a second derivative for the convex function. For the properties and applications of skew-symmetric bifunctions, the reader may consult Antipin [20].

Definition 2.2 (*[19], See also [18]*). Let $T, A: H\to CB(H)$ be set-valued mappings and $N,\eta: H\times H\to H$ be single-valued mappings.

(i) $N(\cdot,\cdot)$ is said to be partially relaxed η-strongly monotone in the first argument with respect to T if there exists a constant $\alpha>0$ such that

$$\langle N(u_1,\cdot)-N(u_2,\cdot), \eta(z,y)\rangle\geq-\alpha\|x-z\|^2, \quad \forall x,y,z\in H, u_1\in T(x), u_2\in T(y).$$

Similarly, we can define the partially relaxed η-strong monotonicity of $N(\cdot,\cdot)$ in the second argument with respect to A.

(ii) $N(\cdot,\cdot)$ is said to be η-strongly monotone in the first argument with respect to T if there exists a constant $\lambda>0$ such that

$$\langle N(u_1,\cdot)-N(u_2,\cdot), \eta(x,y)\rangle\geq\lambda\|x-y\|^2, \quad \forall x,y\in H, u_1\in T(x), u_2\in T(y).$$

(iii) $N(\cdot,\cdot)$ is said to be η-cocoercive in the first argument with respect to T if there exists a constant $\upsilon>0$ such that

$$\langle N(u_1,\cdot)-N(u_2,\cdot), \eta(x,y)\rangle\geq\upsilon\|N(u_1,\cdot)-N(u_2,\cdot)\|^2, \quad \forall x,y\in H, u_1\in T(x), u_2\in T(y).$$

(iv) T is said to be $\tilde{H}$-continuous at $x_0\in H$ if for each $\varepsilon>0$, there exists a neighborhood $N(x_0)$ of x_0 such that

$$\tilde{H}(T(x),T(x_0))\leq\varepsilon, \quad \forall x\in N(x_0).$$

If $N(p,q)=p$ for all $p\in Tx, q\in Ay$ and $\eta(x,y)=x-y, \forall x,y\in H$, then the concept in (i) reduces to the concept of partially relaxed monotonicity of Verma [15] and Noor [10–12]. If $N(p,q)=p$ for all $p\in Tx$,

$q \in Ay\, \forall x, y \in H$, then the concepts in (ii) and (iii) reduce to the concepts of η-strong monotonicity and η-cocoerciveness due to Ansari and Yao [9]. We remark that if $z = x$ in (i), then the partially relaxed η-strong monotonicity is exactly the η-monotonicity for mappings. It is known that the cocoerciveness implies the partially relaxed strong monotonicity, but the converse is not true; see [10–12].

3. Iterative algorithm and convergence

In this section, by using the auxiliary principle technique of Glowinski et al. [1], a three-step iterative algorithm for solving the SGMQVLI (2.1) is suggested and analyzed. The convergence of the iterative sequences generated by the algorithm is proved.

For given $x \in H, u \in T(x)$ and $v \in A(x)$, we consider the following auxiliary variational inclusion problem (AVIP): find $\hat{x} \in H$ such that

$$\langle \hat{x} - x, y - \hat{x} \rangle + \langle \mu N(u, v), \eta(y, \hat{x}) \rangle + \mu\varphi(y, \hat{x}) - \mu\varphi(\hat{x}, \hat{x}) \ge 0, \quad \forall y \in H, \tag{3.1}$$

where $\mu > 0$ is a constant.

We observe that if $\hat{x} = x, \hat{u} \in T(\hat{x})$ and $\hat{v} \in A(\hat{x})$, then $(\hat{x}, \hat{u}, \hat{v})$ is a solution of the GMQVLIP (2.5). By the observation, we can suggest the following three-step iterative algorithm for solving the SGMQVLI (2.1).

Algorithm 3.1. For given $x_0 \in H, u_0 \in T(x_0)$ and $v_0 \in A(x_0)$, compute the approximate solution $(x_n, u_n, \hat{u}_n, \bar{u}_n, v_n, \hat{v}_n, \bar{v}_n)$ of the SGMQVLI (2.1) by the following iterative schemes:

$$\langle y_n - x_n, y - y_n \rangle + \langle \mu N(u_n, v_n), \eta(y, y_n) \rangle + \mu\varphi(y, y_n) - \mu\varphi(y_n, y_n) \ge 0, \quad \forall y \in H, \tag{3.2}$$

$$\langle z_n - y_n, y - z_n \rangle + \langle \beta \hat{N}(\hat{u}_n, \hat{v}_n), \hat{\eta}(y, z_n) \rangle + \beta\hat{\varphi}(y, z_n) - \beta\hat{\varphi}(z_n, z_n) \ge 0, \quad \forall y \in H, \tag{3.3}$$

$$\langle x_{n+1} - z_n, y - x_{n+1} \rangle + \langle \rho \bar{N}(\bar{u}_n, \bar{v}_n), \bar{\eta}(y, x_{n+1}) \rangle + \rho\bar{\varphi}(y, x_{n+1}) - \rho\bar{\varphi}(x_{n+1}, x_{n+1}) \ge 0, \quad \forall y \in H, \tag{3.4}$$

$$\begin{cases} u_n \in T(x_n), & \|u_{n+1} - u_n\| \le (1 + 1/(n+1))\tilde{H}(T(x_{n+1}), T(x_n)), \\ v_n \in A(x_n), & \|v_{n+1} - v_n\| \le (1 + 1/(n+1))\tilde{H}(A(x_{n+1}), A(x_n)), \\ \hat{u}_n \in \hat{T}(y_n), & \|\hat{u}_{n+1} - \hat{u}_n\| \le (1 + 1/(n+1))\tilde{H}(\hat{T}(y_{n+1}), \hat{T}(y_n)), \\ \hat{v}_n \in \hat{A}(y_n), & \|\hat{v}_{n+1} - \hat{v}_n\| \le (1 + 1/(n+1))\tilde{H}(\hat{A}(y_{n+1}), \hat{A}(y_n)), \\ \bar{u}_n \in \bar{T}(z_n), & \|\bar{u}_{n+1} - \bar{u}_n\| \le (1 + 1/(n+1))\tilde{H}(\bar{T}(z_{n+1}), \bar{T}(z_n)), \\ \bar{v}_n \in \bar{A}(z_n), & \|\bar{v}_{n+1} - \bar{v}_n\| \le (1 + 1/(n+1))\tilde{H}(\bar{A}(z_{n+1}), \bar{A}(z_n)), \quad n = 0, 1, 2, \ldots, \end{cases} \tag{3.5}$$

where $\mu > 0$, $\beta > 0$, and $\rho > 0$ are constants.

Lemma 3.1. *Let $(x, u, \hat{u}, \bar{u}, v, \hat{v}, \bar{v})$ be an exact solution of the SGMQVLI (2.1) and $\{x_n\}, \{u_n\}, \{\hat{u}_n\}, \{\bar{u}_n\}, \{v_n\}, \{\hat{v}_n\}$ and $\{\bar{v}_n\}$ be the sequences of approximate solutions of the SGMQVLI (2.1) generated by Algorithm 3.1. Suppose that $\varphi(\cdot,\cdot), \hat{\varphi}(\cdot,\cdot)$ and $\bar{\varphi}(\cdot,\cdot)$ are skew-symmetric bifunctions and $\eta(x, y) = -\eta(y, x)$, $\hat{\eta}(x, y) = -\hat{\eta}(y, x)$, and $\bar{\eta}(x, y) = -\bar{\eta}(y, x)$ for all $x, y \in H$. Assume $N(\cdot,\cdot)$ is partially relaxed η-strongly monotone in the first and second arguments with respect to T and A with constants $\alpha > 0$ and $\gamma > 0$, respectively; $\hat{N}(\cdot,\cdot)$ is partially relaxed $\hat{\eta}$-strongly monotone in the first and second arguments with respect to $\hat{T}$ and $\hat{A}$ with constants $\hat{\alpha} > 0$ and $\hat{\gamma} > 0$, respectively; and $\bar{N}(\cdot,\cdot)$ is partially relaxed $\bar{\eta}$-strongly monotone in the first and second arguments with respect to $\bar{T}$ and $\bar{A}$ with constants $\bar{\alpha} > 0$ and $\bar{\gamma} > 0$, respectively. Then*

$$\|x_{n+1} - x\|^2 \le \|x_n - x\|^2 - (1 - 2\rho(\bar{\alpha} + \bar{\gamma}))\|x_{n+1} - z_n\|^2, \tag{3.6}$$

$$\|z_n - x\|^2 \le \|z_{n-1} - x\|^2 - (1 - 2\beta(\hat{\alpha} + \hat{\gamma}))\|z_n - y_n\|^2, \tag{3.7}$$

$$\|y_n - x\|^2 \le \|y_{n-1} - x\|^2 - (1 - 2\mu(\alpha + \gamma))\|y_n - x_n\|^2. \tag{3.8}$$

Proof. Let $(x, u, \hat{u}, \bar{u}, v, \hat{v}, \bar{v})$ be a solution of the SGMQVLI (2.1). Then $u \in T(x), \hat{u} \in \hat{T}(x), \bar{u} \in \bar{T}(x), v \in A(x), \hat{v} \in \hat{A}(x), \bar{v} \in \bar{A}(x)$ and

$$\langle \mu N(u, v), \eta(y, x) \rangle + \mu\varphi(y, x) - \mu\varphi(x, x) \ge 0, \quad \forall y \in H, \tag{3.9}$$

$$\langle \beta \hat{N}(\hat{u}, \hat{v}), \hat{\eta}(y, x) \rangle + \beta\hat{\varphi}(y, x) - \beta\hat{\varphi}(x, x) \ge 0, \quad \forall y \in H, \tag{3.10}$$

$$\langle \rho \bar{N}(\bar{u}, \bar{v}), \bar{\eta}(y, x) \rangle + \rho\bar{\varphi}(y, x) - \rho\bar{\varphi}(x, x) \ge 0, \quad \forall y \in H, \tag{3.11}$$

where $\mu > 0$, $\beta > 0$ and $\rho > 0$ are constants.

Taking $y = x_{n+1}$ in (3.11) and $y = x$ in (3.4), we have

$$\langle \rho\bar{N}(\bar{u},\bar{v}), \bar{\eta}(x_{n+1},x)\rangle + \rho\bar{\varphi}(x_{n+1},x) - \rho\bar{\varphi}(x,x) \geq 0, \tag{3.12}$$

$$\langle x_{n+1} - z_n, x - x_{n+1}\rangle + \langle \rho\bar{N}(\bar{u}_n,\bar{v}_n), \bar{\eta}(x,x_{n+1})\rangle + \rho\bar{\varphi}(x,x_{n+1}) - \rho\bar{\varphi}(x_{n+1},x_{n+1}) \geq 0. \tag{3.13}$$

Note that $\bar{\varphi}(\cdot,\cdot)$ is skew-symmetric and $\bar{\eta}(x,y) = -\bar{\eta}(y,x)$, $\forall x, y \in H$. Adding (3.12) and (3.13), we get

$$\begin{aligned}\langle x_{n+1} - z_n, x - x_{n+1}\rangle &\geq \rho\langle \bar{N}(\bar{u}_n,\bar{v}_n) - \bar{N}(\bar{u},\bar{v}), \bar{\eta}(x_{n+1},x)\rangle + \rho(\bar{\varphi}(x,x) \\ &\quad - \bar{\varphi}(x_{n+1},x) - \bar{\varphi}(x,x_{n+1}) + \bar{\varphi}(x_{n+1},x_{n+1})) \\ &= \rho\langle \bar{N}(\bar{u}_n,\bar{v}_n) - \bar{N}(\bar{u},\bar{v}_n), \bar{\eta}(x_{n+1},x)\rangle + \rho\langle \bar{N}(\bar{u},\bar{v}_n) - \bar{N}(\bar{u},\bar{v}), \bar{\eta}(x_{n+1},x)\rangle \\ &\geq -\rho(\bar{\alpha}+\bar{\gamma})\|x_{n+1} - z_n\|^2,\end{aligned} \tag{3.14}$$

where we have used the assumption that $\bar{N}(\cdot,\cdot)$ is partially relaxed $\bar{\eta}$-strongly monotone in the first and second arguments with respect to $\bar{T}$ and $\bar{A}$ with constants $\bar{\alpha} > 0$ and $\bar{\gamma} > 0$, respectively. Since

$$\begin{aligned}\|x - z_n\|^2 &= \|x - x_{n+1} + x_{n+1} - z_n\|^2 \\ &= \|x_{n+1} - x\|^2 + \|x_{n+1} - z_n\|^2 + 2\langle x_{n+1} - z_n, x - x_{n+1}\rangle,\end{aligned}$$

it follows from (3.14) that

$$\begin{aligned}\langle x_{n+1} - z_n, x - x_{n+1}\rangle &= \frac{1}{2}[\|x - z_n\|^2 - \|x_{n+1} - x\|^2 - \|x_{n+1} - z_n\|^2] \\ &\geq -\rho(\bar{\alpha}+\bar{\gamma})\|x_{n+1} - z_n\|^2.\end{aligned}$$

Therefore, we get that for $\rho < 1/(2(\bar{\alpha}+\bar{\gamma}))$,

$$\|x_{n+1} - x\|^2 \leq \|z_n - x\|^2 - (1 - 2\rho(\bar{\alpha}+\bar{\gamma}))\|x_{n+1} - z_n\|^2 \leq \|z_n - x\|^2. \tag{3.15}$$

Taking $y = z_n$ in (3.10) and $y = x$ in (3.3), we have

$$\langle \beta\hat{N}(\hat{u},\hat{v}), \hat{\eta}(z_n,x)\rangle + \beta\hat{\varphi}(z_n,x) - \beta\hat{\varphi}(x,x) \geq 0, \tag{3.16}$$

$$\langle z_n - y_n, x - z_n\rangle + \langle \beta\hat{N}(\hat{u}_n,\hat{v}_n), \hat{\eta}(x,z_n)\rangle + \beta\hat{\varphi}(x,z_n) - \beta\hat{\varphi}(z_n,z_n) \geq 0. \tag{3.17}$$

Note that $\hat{\varphi}(\cdot,\cdot)$ is skew-symmetric and $\hat{\eta}(x,y) = -\hat{\eta}(y,x)$, $\forall x, y \in H$. Adding (3.16) and (3.17), we get

$$\begin{aligned}\langle z_n - y_n, x - z_n\rangle &\geq \beta\langle \hat{N}(\hat{u}_n,\hat{v}_n) - \hat{N}(\hat{u},\hat{v}), \hat{\eta}(z_n,x)\rangle \\ &\quad + \beta(\hat{\varphi}(x,x) - \hat{\varphi}(z_n,x) - \hat{\varphi}(x,z_n) + \hat{\varphi}(z_n,z_n)) \\ &\geq \beta\langle \hat{N}(\hat{u}_n,\hat{v}_n) - \hat{N}(\hat{u},\hat{v}_n), \hat{\eta}(z_n,x)\rangle + \beta\langle \hat{N}(\hat{u},\hat{v}_n) - \hat{N}(\hat{u},\hat{v}), \hat{\eta}(z_n,x)\rangle \\ &\geq -\beta(\hat{\alpha}+\hat{\gamma})\|z_n - y_n\|^2,\end{aligned} \tag{3.18}$$

where we have used the assumption that $\hat{N}(\cdot,\cdot)$ is partially relaxed $\hat{\eta}$-strongly monotone in the first and second arguments with respect to $\hat{T}$ and $\hat{A}$ with constants $\hat{\alpha} > 0$ and $\hat{\gamma} > 0$, respectively. Since

$$\begin{aligned}\|x - y_n\|^2 &= \|x - z_n + z_n - y_n\|^2 \\ &= \|z_n - x\|^2 + \|z_n - y_n\|^2 + 2\langle z_n - y_n, x - z_n\rangle,\end{aligned}$$

it follows from (3.18) that

$$\begin{aligned}\langle z_n - y_n, x - z_n\rangle &= \frac{1}{2}[\|y_n - x\|^2 - \|z_n - x\|^2 - \|z_n - y_n\|^2] \\ &\geq -\beta(\hat{\alpha}+\hat{\gamma})\|z_n - y_n\|^2.\end{aligned}$$

Therefore, we get that for $\beta < 1/(2(\hat{\alpha}+\hat{\gamma}))$,

$$\|z_n - x\|^2 \leq \|y_n - x\|^2 - (1 - 2\beta(\hat{\alpha}+\hat{\gamma}))\|z_n - y_n\|^2 \leq \|y_n - x\|^2. \tag{3.19}$$

Taking $y = y_n$ in (3.9) and $y = x$ in (3.2), we have

$$\langle \mu N(u, v), \eta(y_n, x)\rangle + \mu\varphi(y_n, x) - \mu\varphi(x, x) \geq 0, \tag{3.20}$$

$$\langle y_n - x_n, x - y_n\rangle + \langle \mu N(u_n, v_n), \eta(x, y_n)\rangle + \mu\varphi(x, y_n) - \mu\varphi(y_n, y_n) \geq 0. \tag{3.21}$$

Note that $\varphi(\cdot, \cdot)$ is skew-symmetric and $\eta(x, y) = -\eta(y, x), \forall x, y \in H$. Adding (3.20) and (3.21), we get

$$\begin{aligned}\langle y_n - x_n, x - y_n\rangle &\geq \mu\langle N(u_n, v_n) - N(u, v), \eta(y_n, x)\rangle \\ &\quad + \mu(\varphi(x, x) - \varphi(y_n, x) - \varphi(x, y_n) + \varphi(y_n, y_n)) \\ &\geq \mu\langle N(u_n, v_n) - N(u, v_n), \eta(y_n, x)\rangle + \mu\langle N(u, v_n) - N(u, v), \eta(y_n, x)\rangle \\ &\geq -\mu(\alpha + \gamma)\|y_n - x_n\|^2,\end{aligned} \tag{3.22}$$

where we have used the assumption that $N(\cdot, \cdot)$ is partially relaxed η-strongly monotone in the first and second arguments with respect to T and A with constants $\alpha > 0$ and $\gamma > 0$, respectively. Since

$$\begin{aligned}\|x - x_n\|^2 &= \|x - y_n + y_n - x_n\|^2 \\ &= \|y_n - x\|^2 + \|y_n - x_n\|^2 + 2\langle y_n - x_n, x - y_n\rangle,\end{aligned}$$

it follows from (3.22) that

$$\begin{aligned}\langle y_n - x_n, x - y_n\rangle &= \frac{1}{2}[\|x_n - x\|^2 - \|y_n - x\|^2 - \|y_n - x_n\|^2] \\ &\geq -\mu(\alpha + \gamma)\|y_n - x_n\|^2.\end{aligned}$$

Therefore, we get that for $\mu < 1/(2(\alpha + \gamma))$,

$$\|y_n - x\|^2 \leq \|x_n - x\|^2 - (1 - 2\mu(\alpha + \gamma))\|y_n - x_n\|^2 \leq \|x_n - x\|^2. \tag{3.23}$$

Combining (3.15), (3.19) and (3.23), it is easy to see that the conclusions (3.6)–(3.8) hold. □

Now we denote the solution set Sol(2.1) of the SGMQVLI (2.1) as follows:

$$\begin{aligned}\mathrm{Sol}(2.1) = \{&(x, u, \hat{u}, \bar{u}, v, \hat{v}, \bar{v}) \in H \times H \times H \times H \times H \times H \times H : \\ &u \in T(x), \hat{u} \in \hat{T}(x), \bar{u} \in \bar{T}(x), v \in A(x), \hat{v} \in \hat{A}(x), \bar{v} \in \bar{A}(x) \text{and (2.1) holds}\}.\end{aligned}$$

Theorem 3.1. *Let H be a finite-dimensional Hilbert space, $T, \hat{T}, \bar{T}, A, \hat{A}, \bar{A} : H \to C(H)$ be $\tilde{H}$-continuous set-valued mappings and $N, \hat{N}, \bar{N}, \eta, \hat{\eta}, \bar{\eta} : H \times H \to H$ be continuous single-valued mappings such that $\eta(x, y) = -\eta(y, x), \hat{\eta}(x, y) = -\hat{\eta}(y, x)$ and $\bar{\eta}(x, y) = -\bar{\eta}(y, x)$ for all $x, y \in H$. Let $\varphi(\cdot, \cdot), \hat{\varphi}(\cdot, \cdot), \bar{\varphi}(\cdot, \cdot) : H \times H \to R \cup \{+\infty\}$ be continuous skew-symmetric bifunctions. Suppose that $N(\cdot, \cdot)$ is partially relaxed η-strongly monotone in the first and second arguments with respect to T and A with constants $\alpha > 0$ and $\gamma > 0$, respectively; that $\hat{N}(\cdot, \cdot)$ is partially relaxed $\hat{\eta}$-strongly monotone in the first and second arguments with respect to $\hat{T}$ and $\hat{A}$ with constants $\hat{\alpha} > 0$ and $\hat{\gamma} > 0$, respectively, and that $\bar{N}(\cdot, \cdot)$ is partially relaxed $\bar{\eta}$-strongly monotone in the first and second arguments with respect to $\bar{T}$ and $\bar{A}$ with constants $\bar{\alpha} > 0$ and $\bar{\gamma} > 0$ respectively. If the solution set Sol (2.1) of the SGMQVLI (2.1) is nonempty, then for any given $x_0 \in H, u_0 \in T(x_0)$ and $v_0 \in A(x_0)$ the iterative sequences $\{x_n\}, \{u_n\}, \{\hat{u}_n\}, \{\bar{u}_n\}, \{v_n\}, \{\hat{v}_n\}$ and $\{\bar{v}_n\}$ defined by Algorithm 3.1 with $0 < \mu < 1/(2(\alpha+\gamma)), 0 < \beta < 1/(2(\hat{\alpha}+\hat{\gamma}))$ and $0 < \rho < 1/(2(\bar{\alpha} + \bar{\gamma}))$ converge strongly to a solution $(x_*, u_*, \hat{u}_*, \bar{u}_*, v_*, \hat{v}_*, \bar{v}_*)$ of the SGMQVLI (2.1).*

Proof. For any $(x, u, \hat{u}, \bar{u}, v, \hat{v}, \bar{v}) \in$ Sol (2.1), from (3.6)–(3.8) in Lemma 3.1 it follows that the sequences $\{\|x_{n+1} - x\|\}, \{\|z_n - x\}$ and $\{\|y_n - x\|\}$ are nonincreasing and hence $\{x_n\}, \{z_n\}$ and $\{y_n\}$ are bounded. Furthermore, we have

$$\sum_{n=0}^{\infty}(1 - 2\rho(\bar{\alpha} + \bar{\gamma}))\|x_{n+1} - z_n\|^2 \leq \|x_0 - x\|^2,$$

$$\sum_{n=0}^{\infty}(1 - 2\beta(\hat{\alpha} + \hat{\gamma}))\|z_n - y_n\|^2 \leq \|z_0 - x\|^2,$$

$$\sum_{n=0}^{\infty}(1-2\mu(\alpha+\gamma))\|y_n-x_n\|^2 \le \|y_0-x\|^2.$$

These inequalities imply that $\|x_{n+1}-z_n\| \to 0$, $\|z_n-y_n\| \to 0$ and $\|y_n-x_n\| \to 0$ as $n \to \infty$. Therefore we have

$$\|x_{n+1}-x_n\| \le \|x_{n+1}-z_n\| + \|z_n-y_n\| + \|y_n-x_n\| \to 0 \quad \text{as } n \to \infty.$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to x_*$, and hence we have $y_{n_i} \to x_*$ and $z_{n_i} \to x_*$. Since T and A are $\tilde{H}$-continuous on H, by Proposition 1.5.2 of Aubin and Cellina [16, p. 66], T and A are both upper semicontinuous on H. Note that $u_n \in T(x_n)$ and $v_n \in A(x_n)$ for all $n = 0, 1, \ldots$, it follows from Proposition 11.11 of Border [17, p. 57] that there exist a subsequence $\{u_{n_{i_j}}\}$ of $\{u_{n_i}\}$ and a subsequence $\{v_{n_{i_j}}\}$ of $\{v_{n_i}\}$ such that $u_{n_{i_j}} \to u_*$, $v_{n_{i_j}} \to v_*$, $u_* \in T(x_*)$ and $v_* \in A(x_*)$, respectively. By (3.2), we have

$$\langle y_{n_{i_j}} - x_{n_{i_j}}, y - y_{n_{i_j}}\rangle + \langle \mu N(u_{n_{i_j}}, v_{n_{i_j}}), \eta(y, y_{n_{i_j}})\rangle + \mu\varphi(y, y_{n_{i_j}}) - \mu\varphi(y_{n_{i_j}}, y_{n_{i_j}}) \ge 0, \quad \forall y \in H. \tag{3.24}$$

By the continuity of $N(\cdot,\cdot)$ and $\eta(\cdot,\cdot)$ and φ, letting $j \to \infty$ in (3.24), we obtain

$$\langle N(u_*, v_*), \eta(y, x_*)\rangle + \varphi(y, x_*) - \varphi(x_*, x_*) \ge 0, \quad \forall y \in H. \tag{*}$$

Since $\hat{T}$ and $\hat{A}$ are $\tilde{H}$-continuous on H, by Proposition 1.5.2 of Aubin and Cellina [16, p. 66], $\hat{T}$ and $\hat{A}$ are both upper semicontinuous on H. Note that $\hat{u}_n \in \hat{T}(y_n)$ and $\hat{v}_n \in \hat{A}(y_n)$ for all $n = 0, 1, \ldots$, it follows from Proposition 11.11 of Border [17, p. 57] that there exists a subsequence $\{\hat{u}_{n_{i_l}}\}$ of $\{\hat{u}_{n_i}\}$ and a subsequence $\{\hat{v}_{n_{i_l}}\}$ of $\{\hat{v}_{n_i}\}$ such that $\hat{u}_{n_{i_l}} \to \hat{u}_*$, $\hat{v}_{n_{i_l}} \to \hat{v}_*$, $\hat{u}_* \in \hat{T}(x_*)$ and $\hat{v}_* \in \hat{A}(x_*)$, respectively. By (3.3), we have

$$\langle z_{n_{i_l}} - y_{n_{i_l}}, y - z_{n_{i_l}}\rangle + \langle \beta\hat{N}(\hat{u}_{n_{i_l}}, \hat{v}_{n_{i_l}}), \hat{\eta}(y, z_{n_{i_l}})\rangle + \beta\hat{\varphi}(y, z_{n_{i_l}}) - \beta\hat{\varphi}(z_{n_{i_l}}, z_{n_{i_l}}) \ge 0, \quad \forall y \in H. \tag{3.25}$$

By the continuity of $\hat{N}(\cdot,\cdot)$ and $\hat{\eta}(\cdot,\cdot)$ and $\hat{\varphi}$, letting $l \to \infty$ in (3.25), we obtain

$$\langle \hat{N}(\hat{u}_*, \hat{v}_*), \hat{\eta}(y, x_*)\rangle + \hat{\varphi}(y, x_*) - \hat{\varphi}(x_*, x_*) \ge 0, \quad \forall y \in H. \tag{**}$$

Since $\bar{T}$ and $\bar{A}$ are $\tilde{H}$-continuous on H, by Proposition 1.5.2 of Aubin and Cellina [16, p. 66], $\bar{T}$ and $\bar{A}$ are both upper semicontinuous on H. Note that $\bar{u}_n \in \bar{T}(z_n)$ and $\bar{v}_n \in \bar{A}(z_n)$ for all $n = 0, 1, \ldots$, it follows from Proposition 11.11 of Border [17, p. 57] that there exists a subsequence $\{\bar{u}_{n_{i_k}}\}$ of $\{\bar{u}_{n_i}\}$ and a subsequence $\{\bar{v}_{n_{i_k}}\}$ of $\{\bar{v}_{n_i}\}$ such that $\bar{u}_{n_{i_k}} \to \bar{u}_*$, $\bar{v}_{n_{i_k}} \to \bar{v}_*$, $\bar{u}_* \in \bar{T}(x_*)$ and $\bar{v}_* \in \bar{A}(x_*)$, respectively. By (3.4), we have

$$\begin{aligned} &\langle x_{n_{i_k}+1} - z_{n_{i_k}}, y - x_{n_{i_k}+1}\rangle + \langle \rho\bar{N}(\bar{u}_{n_{i_k}}, \bar{v}_{n_{i_k}}), \bar{\eta}(y, x_{n_{i_k}+1})\rangle \\ &\quad + \rho\bar{\varphi}(y, x_{n_{i_k}+1}) - \rho\bar{\varphi}(x_{n_{i_k}+1}, x_{n_{i_k}+1}) \ge 0, \quad \forall y \in H. \end{aligned} \tag{3.26}$$

By the continuity of $\bar{N}(\cdot,\cdot)$ and $\bar{\eta}(\cdot,\cdot)$ and $\bar{\varphi}$, letting $k \to \infty$ in (3.26), we obtain

$$\langle \bar{N}(\bar{u}_*, \bar{v}_*), \bar{\eta}(y, x_*)\rangle + \bar{\varphi}(y, x_*) - \bar{\varphi}(x_*, x_*) \ge 0, \quad \forall y \in H. \tag{***}$$

Now, combining (*), (**) and (***) implies that $(x_*, u_*, \hat{u}_*, \bar{u}_*, v_*, \hat{v}_*, \bar{v}_*)$ is a solution of the SGMQVLI (2.1). Since (3.6)–(3.8) in Lemma 3.1 hold for any $(x, u, \hat{u}, \bar{u}, v, \hat{v}, \bar{v}) \in \mathrm{Sol}$ (2.1), we get

$$\begin{aligned} &\|x_{n+1}-x_*\| \le \|x_n-x_*\|, \quad \forall n = 0, 1, 2, \ldots, \\ &\|z_n-x_*\| \le \|z_{n-1}-x_*\|, \quad \forall n = 0, 1, 2, \ldots, \\ &\|y_n-x_*\| \le \|y_{n-1}-x_*\|, \quad \forall n = 0, 1, 2, \ldots, \end{aligned}$$

which hence imply that $x_n \to x_*$, $z_n \to x_*$ and $y_n \to x_*$ as $n \to \infty$. Since $T, \hat{T}, \bar{T}, A, \hat{A}$ and $\bar{A}$ are $\tilde{H}$-continuous on H, by (3.5) we deduce that as $n \to \infty$,

$$\begin{aligned} &\|u_{n+1}-u_n\| \le (1+1/(n+1))\tilde{H}(T(x_{n+1}), T(x_n)) \to 0, \\ &\|v_{n+1}-v_n\| \le (1+1/(n+1))\tilde{H}(A(x_{n+1}), A(x_n)) \to 0, \\ &\|\hat{u}_{n+1}-\hat{u}_n\| \le (1+1/(n+1))\tilde{H}(\hat{T}(y_{n+1}), \hat{T}(y_n)) \to 0, \\ &\|\hat{v}_{n+1}-\hat{v}_n\| \le (1+1/(n+1))\tilde{H}(\hat{A}(y_{n+1}), \hat{A}(y_n)) \to 0, \end{aligned}$$

$$\|\bar{u}_{n+1} - \bar{u}_n\| \le (1 + 1/(n+1))\tilde{H}(\bar{T}(z_{n+1}), \bar{T}(z_n)) \to 0,$$
$$\|\bar{v}_{n+1} - \bar{v}_n\| \le (1 + 1/(n+1))\tilde{H}(\bar{A}(z_{n+1}), \bar{A}(z_n)) \to 0.$$

Consequently, by using Ding's technique [19, p. 11] we infer that as $n \to \infty$,

$$\|u_n - u_*\| \le \|u_n - u_{n+1}\| + \|u_{n+1} - u_{n+2}\| + \cdots + \|u_{n_{i_j}-1} - u_{n_{i_j}}\| + \|u_{n_{i_j}} - u_*\| \to 0,$$
$$\|\hat{u}_n - \hat{u}_*\| \le \|\hat{u}_n - \hat{u}_{n+1}\| + \|\hat{u}_{n+1} - \hat{u}_{n+2}\| + \cdots + \|\hat{u}_{n_{i_l}-1} - \hat{u}_{n_{i_l}}\| + \|\hat{u}_{n_{i_l}} - \hat{u}_*\| \to 0,$$
$$\|\bar{u}_n - \bar{u}_*\| \le \|\bar{u}_n - \bar{u}_{n+1}\| + \|\bar{u}_{n+1} - \bar{u}_{n+2}\| + \cdots + \|\bar{u}_{n_{i_k}-1} - \bar{u}_{n_{i_k}}\| + \|\bar{u}_{n_{i_k}} - \bar{u}_*\| \to 0,$$

i.e. $u_n \to u_*$, $\hat{u}_n \to \hat{u}_*$ and $\bar{u}_n \to \bar{u}_*$ as $n \to \infty$. Similarly, we can prove that $v_n \to v_*$, $\hat{v}_n \to \hat{v}_*$ and $\bar{v}_n \to \bar{v}_*$ as $n \to \infty$. □

Remark 3.1. If $\varphi = \hat{\varphi} = \bar{\varphi}, T = \hat{T} = \bar{T}, A = \hat{A} = \bar{A}, N = \hat{N} = \bar{N}, \eta = \hat{\eta} = \bar{\eta}$, and $\mu = \beta = \rho$, then Theorem 3.1 reduces to Ding's Theorem 3.1 [19]. We emphasize that the set-valued mappings $T, \hat{T}, \bar{T}, A, \hat{A}$ and $\bar{A}$ may not be Lipschitz continuous in Theorem 3.1. Hence Theorem 3.1 improves, generalizes, and unifies Ding's Theorem 3.1 [19], and the corresponding results in [10–13,18].

Remark 3.2. We observe that if ϕ, $\hat{\phi}$ and $\bar{\phi}$ in Theorem 3.1 are functions of one variable, then the continuity assumption can be replaced by the assumption of lower semicontinuity.

4. Special cases of Algorithm 3.1

We have seen in Section 2 that with appropriate restrictions on the mappings and/or parameters, our system of generalized mixed quasi-variational-like inclusions reduces to various systems of inclusions known in the literature. In this section, we shall illustrate our Algorithm 3.1 and extend and unify some known algorithms in the literature as well.

4.1. Special case

If $\varphi = \hat{\varphi} = \bar{\varphi}$, $T = \hat{T} = \bar{T}$, $A = \hat{A} = \bar{A}$, $N = \hat{N} = \bar{N}$, $\eta = \hat{\eta} = \bar{\eta}$, and $\mu = \beta = \rho$, then Algorithm 3.1 reduces to Ding's Algorithm 3.1 [19] for computing the approximate solutions of the GMQVLIP (2.5).

4.2. Special case

If $\eta(x, y) = g(x) - g(y)$, $\hat{\eta}(x, y) = \hat{g}(x) - \hat{g}(y)$, $\bar{\eta}(x, y) = \bar{g}(x) - \bar{g}(y), \forall x, y \in H$ where $g, \hat{g}, \bar{g} : H \to H$ are three given single-valued mappings, then Algorithm 3.1 reduces to the following three-step iterative algorithm for solving the system of generalized mixed quasi-variational inclusions (2.2).

Algorithm 4.1. For given $x_0 \in H, u_0 \in T(x_0)$ and $v_0 \in A(x_0)$, compute the approximate solution $(x_n, u_n, \hat{u}_n, \bar{u}_n, v_n, \hat{v}_n, \bar{v}_n)$ of the $SGMQVI$ (2.2) by the following iterative schemes:

$$\langle y_n - x_n, y - y_n\rangle + \langle \mu N(u_n, v_n), g(y) - g(y_n)\rangle + \mu\varphi(y, y_n) - \mu\varphi(y_n, y_n) \ge 0, \quad \forall y \in H, \tag{4.1}$$

$$\langle z_n - y_n, y - z_n\rangle + \langle \beta\hat{N}(\hat{u}_n, \hat{v}_n), \hat{g}(y) - \hat{g}(z_n)\rangle + \beta\hat{\varphi}(y, z_n) - \beta\hat{\varphi}(z_n, z_n) \ge 0, \quad \forall y \in H, \tag{4.2}$$

$$\langle x_{n+1} - z_n, y - x_{n+1}\rangle + \langle \rho\bar{N}(\bar{u}_n, \bar{v}_n), \bar{g}(y) - \bar{g}(x_{n+1})\rangle + \rho\bar{\varphi}(y, x_{n+1}) - \rho\bar{\varphi}(x_{n+1}, x_{n+1}) \ge 0, \quad \forall y \in H, \tag{4.3}$$

$$\begin{cases} u_n \in T(x_n), & \|u_{n+1} - u_n\| \le (1 + 1/(n+1))\tilde{H}(T(x_{n+1}), T(x_n)), \\ v_n \in A(x_n), & \|v_{n+1} - v_n\| \le (1 + 1/(n+1))\tilde{H}(A(x_{n+1}), A(x_n)), \\ \hat{u}_n \in \hat{T}(y_n), & \|\hat{u}_{n+1} - \hat{u}_n\| \le (1 + 1/(n+1))\tilde{H}(\hat{T}(y_{n+1}), \hat{T}(y_n)), \\ \hat{v}_n \in \hat{A}(y_n), & \|\hat{v}_{n+1} - \hat{v}_n\| \le (1 + 1/(n+1))\tilde{H}(\hat{A}(y_{n+1}), \hat{A}(y_n)), \\ \bar{u}_n \in \bar{T}(z_n), & \|\bar{u}_{n+1} - \bar{u}_n\| \le (1 + 1/(n+1))\tilde{H}(\bar{T}(z_{n+1}), \bar{T}(z_n)), \\ \bar{v}_n \in \bar{A}(z_n), & \|\bar{v}_{n+1} - \bar{v}_n\| \le (1 + 1/(n+1))\tilde{H}(\bar{A}(z_{n+1}), \bar{A}(z_n)), \quad n = 0, 1, 2, \ldots, \end{cases} \tag{4.4}$$

where $\mu > 0$, $\beta > 0$, and $\rho > 0$ are constants.

Furthermore, if $\varphi = \hat{\varphi} = \bar{\varphi}$, $T = \hat{T} = \bar{T}$, $A = \hat{A} = \bar{A}$, $N = \hat{N} = \bar{N}$, $\eta = \hat{\eta} = \bar{\eta}$, and $\mu = \beta = \rho$, then Algorithm 4.1 reduces to Ding's Algorithm 3.2 [19] for computing the approximate solutions of the generalized mixed quasi-variational inclusion problem (2.5) in [19].

4.3. Special case

If $\eta(x, y) = \hat{\eta}(x, y) = \bar{\eta}(x, y) = x - y, \forall x, y \in H$, then Algorithm 3.1 reduces to the following three-step iterative algorithm for solving the system of generalized mixed quasi-variational inclusions (2.3).

Algorithm 4.2. For given $x_0 \in H, u_0 \in T(x_0)$ and $v_0 \in A(x_0)$, compute the approximate solution $(x_n, u_n, \hat{u}_n, \bar{u}_n, v_n, \hat{v}_n, \bar{v}_n)$ to the system of generalized mixed quasi-variational inclusions (2.3) by the following iterative schemes:

$$\langle y_n - x_n + \mu N(u_n, v_n), y - y_n\rangle + \mu\varphi(y, y_n) - \mu\varphi(y_n, y_n) \geq 0, \quad \forall y \in H, \tag{4.5}$$

$$\langle z_n - y_n + \beta\hat{N}(\hat{u}_n, \hat{v}_n), y - z_n\rangle + \beta\hat{\varphi}(y, z_n) - \beta\hat{\varphi}(z_n, z_n) \geq 0, \quad \forall y \in H, \tag{4.6}$$

$$\langle x_{n+1} - z_n + \rho\bar{N}(\bar{u}_n, \bar{v}_n), y - x_{n+1}\rangle + \rho\bar{\varphi}(y, x_{n+1}) - \rho\bar{\varphi}(x_{n+1}, x_{n+1}) \geq 0, \quad \forall y \in H, \tag{4.7}$$

$$\begin{cases} u_n \in T(x_n), & \|u_{n+1} - u_n\| \leq (1 + 1/(n+1))\tilde{H}(T(x_{n+1}), T(x_n)), \\ v_n \in A(x_n), & \|v_{n+1} - v_n\| \leq (1 + 1/(n+1))\tilde{H}(A(x_{n+1}), A(x_n)), \\ \hat{u}_n \in \hat{T}(y_n), & \|\hat{u}_{n+1} - \hat{u}_n\| \leq (1 + 1/(n+1))\tilde{H}(\hat{T}(y_{n+1}), \hat{T}(y_n)), \\ \hat{v}_n \in \hat{A}(y_n), & \|\hat{v}_{n+1} - \hat{v}_n\| \leq (1 + 1/(n+1))\tilde{H}(\hat{A}(y_{n+1}), \hat{A}(y_n)), \\ \bar{u}_n \in \bar{T}(z_n), & \|\bar{u}_{n+1} - \bar{u}_n\| \leq (1 + 1/(n+1))\tilde{H}(\bar{T}(z_{n+1}), \bar{T}(z_n)), \\ \bar{v}_n \in \bar{A}(z_n), & \|\bar{v}_{n+1} - \bar{v}_n\| \leq (1 + 1/(n+1))\tilde{H}(\bar{A}(z_{n+1}), \bar{A}(z_n)), \quad n = 0, 1, 2, \ldots, \end{cases} \tag{4.8}$$

where $\mu > 0$, $\beta > 0$, and $\rho > 0$ are constants.

Furthermore, if $\varphi = \hat{\varphi} = \bar{\varphi}$, $T = \hat{T} = \bar{T}$, $A = \hat{A} = \bar{A}$, $N = \hat{N} = \bar{N}$, and $\mu = \beta = \rho$, then Algorithm 4.2 reduces to Ding's Algorithm 3.3 [19] for computing the approximate solutions of the generalized mixed quasi-variational inclusion problem (2.6) in [19].

4.4. Special case

If φ, $\hat{\varphi}$ and $\bar{\varphi}$ are as in special case (I) of Section 2, then Algorithm 3.1 can be rewritten as follows.

Algorithm 4.3. For given $x_0 \in H$, $u_0 \in T(x_0)$ and $v_0 \in A(x_0)$, the following iterative schemes enable us to compute $(x_n, u_n, \hat{u}_n, \bar{u}_n, v_n, \hat{v}_n, \bar{v}_n)$:

$$y_n = J_\mu^{\Delta\varphi(\cdot, y_n)}[x_n - \mu N(u_n, v_n)], \tag{4.9}$$

$$z_n = J_\beta^{\Delta\hat{\varphi}(\cdot, z_n)}[y_n - \beta\hat{N}(\hat{u}_n, \hat{v}_n)], \tag{4.10}$$

$$x_{n+1} = J_\rho^{\Delta\bar{\varphi}(\cdot, x_{n+1})}[z_n - \rho\bar{N}(\bar{u}_n, \bar{v}_n)], \tag{4.11}$$

$$\begin{cases} u_n \in T(x_n), & \|u_{n+1} - u_n\| \leq (1 + 1/(n+1))\tilde{H}(T(x_{n+1}), T(x_n)), \\ v_n \in A(x_n), & \|v_{n+1} - v_n\| \leq (1 + 1/(n+1))\tilde{H}(A(x_{n+1}), A(x_n)), \\ \hat{u}_n \in \hat{T}(y_n), & \|\hat{u}_{n+1} - \hat{u}_n\| \leq (1 + 1/(n+1))\tilde{H}(\hat{T}(y_{n+1}), \hat{T}(y_n)), \\ \hat{v}_n \in \hat{A}(y_n), & \|\hat{v}_{n+1} - \hat{v}_n\| \leq (1 + 1/(n+1))\tilde{H}(\hat{A}(y_{n+1}), \hat{A}(y_n)), \\ \bar{u}_n \in \bar{T}(z_n), & \|\bar{u}_{n+1} - \bar{u}_n\| \leq (1 + 1/(n+1))\tilde{H}(\bar{T}(z_{n+1}), \bar{T}(z_n)), \\ \bar{v}_n \in \bar{A}(z_n), & \|\bar{v}_{n+1} - \bar{v}_n\| \leq (1 + 1/(n+1))\tilde{H}(\bar{A}(z_{n+1}), \bar{A}(z_n)), \quad n = 0, 1, 2, \ldots, \end{cases} \tag{4.12}$$

where for each $x \in H$, $J_\mu^{\Delta\varphi(\cdot,x)} = (I + \mu\Delta\varphi(\cdot, x))^{-1}$, $J_\beta^{\Delta\hat{\varphi}(\cdot,x)} = (I + \beta\Delta\hat{\varphi}(\cdot, x))^{-1}$, and $J_\rho^{\Delta\bar{\varphi}(\cdot,x)} = (I + \rho\Delta\bar{\varphi}(\cdot, x))^{-1}$ are the η-proximal mapping of $\varphi(\cdot, x)$, the $\hat{\eta}$-proximal mapping of $\hat{\varphi}(\cdot, x)$, and the $\bar{\eta}$-proximal mapping of $\bar{\varphi}(\cdot, x)$, respectively, (see [21]) and $\mu > 0, \beta > 0, \rho > 0$ are constants.

When $\varphi(\cdot,\cdot)$, $\hat{\varphi}(\cdot,\cdot)$, and $\bar{\varphi}(\cdot,\cdot)$ are all proper convex and lower semicontinuous in the first argument on H and $\eta(y,x) = \hat{\eta}(y,x) = \bar{\eta}(y,x) = y - x$ for each $y, x \in H$, Algorithm 4.3 is a three-step forward–backward splitting algorithm for solving the system of generalized mixed quasi-variational inclusions (2.3). Moreover, it is easy to see that Algorithm 3.4 of Ding [19] is a special case of the above Algorithm 4.3.

All in all, Algorithm 3.1 improves and generalizes Algorithms 3.1–3.4 of Ding [18,19] and Algorithms 3.1–3.3 of Noor [10–12] to the system of generalized mixed quasi-variational-like inclusions.

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